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Gravitational F-terms of $\mathcal{N}=1$ supersymmetric gauge theories

Ido Adam and Yaron Oz

Raymond and Beverly Sackler Faculty of Exact Sciences
School of Physics and Astronomy
Tel-Aviv University, Ramat-Aviv 69978, Israel
E-mail: adamido@post.tau.ac.il, yaronoz@post.tau.ac.il

Abstract: We consider four-dimensional $\mathcal{N}=1$ supersymmetric gauge theories in a supergravity background. We use generalized Konishi anomaly equations and R-symmetry anomaly to compute the exact perturbative and non-perturbative gravitational F-terms. We study two types of theories: The first model breaks supersymmetry dynamically, and the second is based on a G_2 gauge group. The results are compared with the corresponding vector models. We discuss the diagrammatic expansion of the G_2 theory.

Keywords: Supersymmetry Breaking, Supersymmetry and Duality, Supersymmetric Effective Theories.

Contents

1. Introduction

F-terms of four-dimensional supersymmetric gauge theories in a supergravity and graviphoton backgrounds have attracted much attention in recent years. On the one hand they are related to certain exactly computable amplitudes of two gravitons and graviphotons. On the other hand they are computed by second quantized partition functions of topological strings [1], and have an interesting mathematical structure [2]. Gravitational F-terms are directly related to the partition function of two-dimensional non-critical strings [3, 4]. Recently, gravitational F-terms have been related to the computation of certain $\mathcal{N} = 2$ black holes partition function [5].

In this paper we will consider the gravitational F-terms in the context of four-dimensional $\mathcal{N} = 1$ supersymmetric gauge theories. Dijkgraaf and Vafa suggested a matrix model description, where the gravitational F-terms can be computed by summing up the non-planar matrix diagrams [6]. The assumption made is that the relevant fields are the glueball superfields S_i and the F-terms are holomorphic couplings of the glueball superfields to gravity. The DV matrix proposal has been proven diagrammatically in [7, 8].

In this paper we will consider the gravitational F-terms of the form

$$\Gamma_G = \int d^4x d^2\theta G_{\alpha\beta\gamma} G^{\alpha\beta\gamma} F_1(S_i)\,, \tag{1.1}$$

where $G_{\alpha\beta\gamma}$ is the $\mathcal{N}=1$ Weyl superfield. According to the DV proposal, $F_1(S_i)$ is the partition function of the corresponding matrix model evaluated by summing the genus one diagrams with S_i being the 't Hooft parameter.

The approach we will take is to use generalized Konishi anomaly equations and R-symmetry anomaly to compute the exact perturbative and non-perturbative gravitational F-terms. We will consider a vanishing graviphoton background. In general, it is not clear in which cases the generalized Konishi anomaly equations are sufficient in order to determine the gravitational F-terms. We will study two types of theories: The first model breaks supersymmetry dynamically, and the second is based on a gauge group that does not have a large N_c expansion. We will consider a G_2 gauge group.

In a model that breaks supersymmetry the chiral ring relations cannot be used. This will be analysed following [9], by adding a certain deformation to the tree-level superpotential. The model based on the G_2 gauge group does not have a large N_c expansion. This complicates the relation between the matrix (vector) model computations and the gauge theory ones. In both cases, we will compare the results to their counterparts in the corresponding vector models. We will also discuss the diagrammatic expansion of the G_2 theory.

The paper is organized as follows. In section 2 the computational scheme for computing the gravitational F-terms is reviewed following [10, 11]. In section 3 the gravitational F-term for the model that breaks supersymmetry dynamically is computed, and compared with the corresponding vector model. In section 4 the same computation and comparison are performed for the G_2 SYM theory. Details of the computations are presented for the two models in appendices A and B respectively. In appendix C we discuss the diagrammatics of the G_2 model.

Other recent works on the computation of gravitational F-terms are [12, 13, 10, 14, 15, 16, 17].

2. The computational scheme

In this section we will review the computational scheme for computing the gravitational F-terms.

2.1 Deformed chiral ring

Consider first an $\mathcal{N}=1$ supersymmetric gauge theory in flat space with a gauge group G and some matter supermultiplets. We will denote the four-dimensional Weyl spinor

JHEP11(2004)071

supersymmetry generators by Q_α and $\bar{Q}_{\dot{\alpha}}$. Chiral operators are operators annihilated by $\bar{Q}_{\dot{\alpha}}$. For instance, the lowest component ϕ of a chiral superfield Φ is a chiral operator. The OPE of two chiral operators is nonsingular and allows for the definition of the product of two chiral operators. The product of chiral operators is also a chiral operator. Furthermore, one can define a ring structure on the set of equivalence classes of chiral operators modulo operators of the form $\{\bar{Q}_{\dot{\alpha}}, \ldots]$.

Denote by V the vector superfield in the adjoint representation of G, by Φ chiral superfields in a representation r of G and by ϕ their lowest component. The field strength (spinor) superfield is $W_\alpha = -\frac{1}{4}\bar{D}^2 e^{-V} D_\alpha e^V$ and is a chiral superfield. One has

$$\{W_\alpha^{(r)}, W_\beta^{(r)}\} = 0\,, \qquad W_\alpha^{(r)}\phi^{(r)} = 0\,, \tag{2.1}$$

modulo $\{\bar{Q}_{\dot{\alpha}}, \ldots]$ terms, where we noted that ϕ transforms in a representation r of the gauge group G, such that $W_\alpha^{(r)} = W_\alpha^a T^a(r)$ with $T^a(r)$ being the generators of the gauge group G in the representation r.

Consider next the coupling of the supersymmetric gauge theory to a background $\mathcal{N}=1$ supergravity. We denote by $G^{\alpha\beta\gamma}$ the $\mathcal{N}=1$ Weyl superfield. In the following we will denote by W_α the supersymmetric gauge field strength as well as its lowest component, the gaugino, and similarly for $G^{\alpha\beta\gamma}$. The chiral ring relations (2.1) are deformed to

$$\left\{W_\alpha^{(r)}, W_\beta^{(r)}\right\} = 2G_{\alpha\beta\gamma}W^{\gamma(r)}\,, \qquad W_\alpha^{(r)}\phi^{(r)} = 0\,. \tag{2.2}$$

Together with Bianchi identities of $\mathcal{N}=1$ supergravity these relations generate all the relations in the deformed chiral ring. Some relations that will be used later are [10]

$$\begin{aligned} [W^2, W_\alpha] = 0\,, \qquad W^2 W_\alpha = -\frac{1}{3}G^2 W_\alpha\,, \qquad W^2W^2 = -\frac{1}{3}G^2W^2\,, \\ G^4 = (G^2)^2 = 0\,. \end{aligned} \tag{2.3}$$

Throughout the paper we will follow the conventions used in [11].

In addition to the above kinematical relations, one has kinematical relations for the matter fields and dynamical relations from the variation of the tree level superpotential W_{tree}

$$\phi\frac{\partial W_{\text{tree}}}{\partial\phi} = 0\,. \tag{2.4}$$

2.2 Konishi anomaly relations

The classical chiral ring relations are, in general, modified quantum mechanically. The classical relations arising from (2.4) have a natural generalization, as anomalous Ward identities of the quantized matter sector in a classical gauge(ino) and supergravity background. The classical Konishi equation reads

$$\bar{D}^2 J = \phi'\frac{\partial W_{\text{tree}}}{\partial\phi}\,, \tag{2.5}$$

where J is the generalized Konishi current and $\delta\phi = \phi'(\phi)$ is the generalized Konishi transformation. This relation gets an anomalous contribution in the quantum theory. It takes the form [18, 19, 20]

$$\bar{D}^2 J = \phi'_i \frac{\partial W_{\text{tree}}}{\partial \phi_i} + \frac{1}{32\pi^2}\left(W_{\alpha i}{}^j W^\alpha{}_j{}^k + \frac{1}{3}G^2\delta_i^k\right)\frac{\partial \phi'_k}{\partial \phi_i}\,, \tag{2.6}$$

where i, j and k are gauge indices and their contraction is in the appropriate representation.

Since the divergence $\bar{D}^2 J$ is $\bar{Q}$-exact it vanishes in a supersymmetric vacuum. Taking the expectation value of (2.6) in a slowly varying gaugino background S, we get the Konishi relations in a supergravity background given by G^2

$$\left\langle \phi'_i \frac{\partial W_{\text{tree}}}{\partial \phi}\right\rangle_S + \left\langle\left(\frac{1}{32\pi^2}W_{\alpha i}{}^j W^\alpha{}_j{}^k + \frac{1}{32\pi^2}\frac{G^2}{3}\delta_i^k\right)\frac{\partial \phi'_k}{\partial \phi_i}\right\rangle_S = 0\,. \tag{2.7}$$

We will use this relation to determine the supergravity corrections to the chiral correlators, which in turn can be integrated to give the perturbative part of the gravitational F-terms of the corresponding $\mathcal{N}=1$ gauge theory. Henceforth, we absorb the factor of $\frac{1}{32\pi^2}$ within G^2.

2.3 Computation of gravitational F-terms

We are interested in the low energy description of a four-dimensional $\mathcal{N}=1$ supersymmetric gauge theory in the background of $\mathcal{N}=1$ supergravity. The assumption is that the relevant field is the glueball superfield S and the F-terms are holomorphic couplings of the glueball superfield to gravity.

In the absence of supergravity, the only relevant F-term is the effective glueball superpotential

$$\Gamma_0 = \int d^4x d^2\theta W_{\text{eff}}(S)\,, \tag{2.8}$$

where

$$S = -\frac{1}{32\pi^2} Tr\, W_\alpha W^\alpha\,. \tag{2.9}$$

In the matrix model description Γ_0 is computed by summing up planar diagrams and adding a non-perturbative Veneziano-Yankielowicz superpotential [21].

When coupled to supergravity there is a gravitational F-term of the form

$$\Gamma_1 = \int d^4x d^2\theta W_1(S) G^2\,. \tag{2.10}$$

In the matrix model description it is computed by summing up non-planar diagrams and adding a non-perturbative contribution. Note that terms with higher powers of G vanish due to the chiral ring relation (2.3).

2.3.1 Computation of $W_1(S)$

Consider the supersymmetric gauge theory with a tree level superpotential

$$W_{\text{tree}} = \sum_I g_I \sigma_I\,, \tag{2.11}$$

where σ_I are gauge invariant chiral operators and g_I the tree level couplings. The gradient equations for the holomorphic part of the effective action read

$$\frac{\partial\left(W_{\rm eff}+G^2W_1\right)}{\partial g_I}=\langle\sigma_I\rangle_S\,. \tag{2.12}$$

The expectation values are taken in a slowly varying (classical) gaugino and gravitino background.

As first discussed in [18], for a gauge theory in the absence of a supergravity background the Konishi relations (2.7) can be used to solve for the expectation values $\langle\sigma_I\rangle_S$ as a function of S and the tree level couplings. One can then integrate (2.12) to determine the dependence of $W_{\rm eff}$ on the tree level couplings. For the gravitational coupling $W_1(S)$ a similar reasoning applies. However, we will have to take into account the effects of the supergravity background on the correlators of chiral operators.

In the absence of gravity the correlators of chiral operators factorize

$$\langle\sigma_I\sigma_J\rangle=\langle\sigma_I\rangle\langle\sigma_J\rangle\,. \tag{2.13}$$

In a matrix model description, corresponding to gauge theories with large N_c expansion, this is the feature of the planar limit. Here and in some of the equations in the following we omit for simplicity the subscript S. Eq. (2.13) can be used in the relations (2.7) in order to solve for $\langle\sigma_I\rangle_S$ as a function of (S,g_I).

However, in the presence of supergravity the chiral correlators do not factorize, and instead we have

$$\langle\sigma_I\sigma_J\rangle=\langle\sigma_I\sigma_J\rangle_c+\langle\sigma_I\rangle\langle\sigma_J\rangle\,, \tag{2.14}$$

with analogous relations for correlators with more chiral operators. Also, the one point functions have to be expanded in G^2 as

$$\langle\sigma_I\rangle=\langle\sigma_I\rangle_1+G^2\langle\sigma_I\rangle_2\,. \tag{2.15}$$

Note that this expansion is exact in the chiral ring due to the fact that G^4 vanishes modulo $\bar{D}$ exact terms. Thus, we have to express $\langle\sigma_I\rangle_1$ and $\langle\sigma_I\rangle_2$ as functions of S and g_I.

In the next section we will show explicitly that there are enough relations (2.7) to solve for $\langle\sigma_I\rangle,\langle\sigma_J\rangle$ as well as for the connected correlators $\langle\sigma_I\sigma_J\rangle_c$. The perturbative part of the gravitational coupling $W_1(S)$ is then obtained by integrating the gravitational contribution $\langle\sigma_K\rangle_2$ in (2.15) for the $\langle\sigma_K\rangle$ appearing in the tree level potential (2.11), with respect to the couplings g_K.

Note, that a crucial ingredient in the analysis is the assumption that connected correlators of three or more chiral operators vanish in the gravitationally deformed chiral ring.

The procedure outlined above determines $W_1(S)$ up to an integration constant independent of the couplings g_I. The integration constant can be determined by the one loop exact $\mathrm{U}(1)_R$ anomaly, as will be done later.

JHEP11(2004)071

3. Dynamical supersymmetry breaking

The model considered is an $\mathcal{N} = 1$ SYM theory with an $\mathrm{Sp}(N_c)$ gauge group coupled to $2N_f = 2(N_c + 1)$ fundamental chiral multiplets Q^i_a ($a = 1, \ldots, 2N_c$ is the gauge index and the flavor index is $i = 1, \ldots, 2N_f$) and a chiral gauge singlet S_{ij} antisymmetric in the flavor indices [22, 23]. The gauge invariant matter in the theory is the S_{ij} and the mesons $M^{ij} = Q^{ai}Q_{aj}$, which are antisymmetric in the flavor indices i and j.

The tree-level superpotential is taken to be

$$W_{\mathrm{tree}} = \lambda S_{ij} M^{ij} - m J^{ij} S_{ij} \,, \tag{3.1}$$

where $J = \mathbf{1}_{N_c} \otimes i\sigma^2$ is the symplectic form.

This theory has no supersymmetric vacuum, so the chiral ring relations cannot be used. This can be remedied following [9], by adding a deformation to the tree-level superpotential giving mass to S_{ij}:

$$W_{\mathrm{tree}} = \lambda S_{ij} M^{ij} - m J^{ij} S_{ij} + \alpha S_{ij} S^{ij} \,, \tag{3.2}$$

where $S^{ij} = S_{kl} J^{ki} J^{lj}$. This deformation adds a supersymmetric vacuum and enables the use of the chiral ring.

3.1 Calculation of W_1

3.1.1 The perturbative superpotential

Using Konishi transformations either in Q^i_a or S_{ij} (the detailed computation is in appendix A) the following Konishi anomaly equations are obtained

$$\begin{aligned}
S\delta^l_i &= 2\lambda \left\langle S_{ij} M^{lj} \right\rangle + \frac{2N_c}{3} G^2 \delta^l_i \,, && (3.3)\\
0 &= \lambda \left\langle S_{lm} M^{ij} \right\rangle - m \left\langle S_{lm} \right\rangle J^{ij} + 2\alpha \left\langle S_{lm} S^{ij} \right\rangle + \frac{1}{6} G^2 (\delta^i_l \delta^j_m - \delta^i_m \delta^j_l) \,, && (3.4)\\
S \left\langle M^n \right\rangle \delta^l_i &= 2\lambda \left\langle S_{ij} M^{lj} M^n \right\rangle + \frac{2N_c}{3} G^2 \left\langle M^n \right\rangle \delta^l_i + \frac{2n}{3} G^2 (J^{-1})_{ij} \left\langle M^{lj} M^{n-1} \right\rangle \,, && (3.5)\\
0 &= \lambda \left\langle S_{ij} M^{lm} \tilde{S}^n \right\rangle - m J^{lm} \left\langle S_{ij} \tilde{S}^n \right\rangle + 2\alpha \left\langle S_{ij} S^{lm} \tilde{S}^n \right\rangle + \\
&\quad + \frac{1}{6} G^2 \left\langle \tilde{S}^n \right\rangle (\delta^l_i \delta^m_j - \delta^l_j \delta^m_i) + \frac{n}{3} G^2 J^{lm} \left\langle S_{ij} \tilde{S}^{n-1} \right\rangle \,, && (3.6)\\
S \left\langle \tilde{S}^n \right\rangle \delta^l_i &= 2\lambda \left\langle S_{ij} M^{lj} \tilde{S}^n \right\rangle + \frac{2N_c}{3} G^2 \left\langle \tilde{S}^n \right\rangle \delta^l_i \,, && (3.7)\\
0 &= \lambda \left\langle S_{ij} M^{lm} M^n \right\rangle - m J^{lm} \left\langle S_{ij} M^n \right\rangle + 2\alpha \left\langle S_{ij} S^{lm} M^n \right\rangle + \\
&\quad + \frac{1}{6} G^2 \left\langle M^n \right\rangle (\delta^l_i \delta^m_j - \delta^l_j \delta^m_i) \,, && (3.8)\\
S \left\langle S_{ki} \right\rangle &= 2\lambda \left\langle S_{kl} M^{lm} S_{im} \right\rangle + \frac{2N_c}{3} G^2 \left\langle S_{ki} \right\rangle \,, && (3.9)
\end{aligned}$$

where $\tilde{S} = J^{ij} S_{ij}$ and $M = J^{ij} M_{ij}$.

Assuming flavor symmetry and that all the connected three-point-functions vanish, these equations can be solved for the correlation functions. Picking the solution corresponding to the massive vacuum, in which the chiral multiplets are massive, the order G^2

terms of the relevant connected correlation functions (denoted by $\langle \ldots \rangle^g$) are

$$\left\langle S_{ij} M^{ij} \right\rangle_c^g = \frac{N_f(2N_f-1)\left(-m^2+4\alpha S+m\sqrt{m^2-4\alpha S}\right)}{6\lambda\,(m^2-4\alpha S)}\,, \tag{3.10}$$

$$\langle S_{ij}\rangle^g = \frac{-4N_c\sqrt{m^2-4\alpha S}+(2N_f-1)\left(-m+\sqrt{m^2-4\alpha S}\right)}{12\,(m^2-4\alpha S)}(J^{-1})_{ij}\,, \tag{3.11}$$

$$\left\langle M^{ij}\right\rangle^g = \frac{\alpha\left[-4N_c\sqrt{m^2-4\alpha S}+(2N_f-1)(-m+\sqrt{m^2-4\alpha S})\right]}{6\lambda(m^2-4\alpha S)}J^{ij}\,, \tag{3.12}$$

$$\left\langle S_{ij} S^{ij} \right\rangle_c^g = -\frac{2N_f(2N_f-1)\left(m^2-4\alpha S+m\sqrt{m^2-4\alpha S}\right)}{24\alpha\,(m^2-4\alpha S)}\,. \tag{3.13}$$

The gradient equations (2.12) in this model read

$$\frac{\partial W_1}{\partial\lambda} = \left\langle S_{ij} M^{ij}\right\rangle^g\,, \tag{3.14}$$

$$\frac{\partial W_1}{\partial m} = -J^{ij}\,\langle S_{ij}\rangle^g\,, \tag{3.15}$$

$$\frac{\partial W_1}{\partial\alpha} = \left\langle S_{ij} S^{ij}\right\rangle^g\,. \tag{3.16}$$

These can be integrated in order to obtain the gravitational F-term up to a function of S, which is independent of the couplings. For $N_f = N_c + 1$ — the case of unbroken supersymmetry — one obtains

$$\begin{aligned} W_1 = \frac{N_f}{6}\Bigg[&(2N_f-3)\log\left(-\frac{4\alpha}{m^2}S\right)-(2N_f-3)\log\left(1+\sqrt{1-\frac{4\alpha}{m^2}S}\right)- \\ &-(2N_f-1)\log\sqrt{1-\frac{4\alpha}{m^2}S}-2\log m-4N_c\log\lambda\Bigg]+C(S)\,. \end{aligned} \tag{3.17}$$

3.1.2 The non-perturbative contribution

Using the appendix of [24], the R-symmetry anomaly for this model is given by

$$\mathcal{A} = -\frac{1}{3}\left[N_c\,(2N_c+1)-\frac{4}{3}N_fN_c-\frac{1}{3}N_f\,(2N_f-1)\right]G^2 \tag{3.18}$$

in the convention $\frac{1}{32\pi^2}G^2 \to G^2$.

The term in the action that reproduces this anomaly is similar to the one in [11]:

$$\begin{aligned} \Gamma_1(S,G^2) = \int d^4x d^2\theta \frac{1}{6}\Bigg[&N_c\,(2N_c+1)\log\frac{S}{\Lambda_1^3}+\frac{4}{3}N_fN_c\log\frac{\Lambda_1}{\mu}+ \\ &+\frac{1}{3}N_f(2N_f-1)\log\frac{\Lambda_1}{\alpha}\Bigg]G^2\,, \end{aligned} \tag{3.19}$$

where Λ_1 is the supergravity scale and μ is the mass scale for the massless matter multiplets in the fundamental representation. The R-symmetry transformation is taken to be

$$\begin{aligned} \theta' &= e^{-i\alpha}\theta\,, \qquad \bar\theta' = e^{i\alpha}\bar\theta\,, \\ S'(x,\theta',\bar\theta') &= e^{-2i\alpha}S(x,\theta,\bar\theta)\,, \\ G'^2(x,\theta',\bar\theta') &= e^{-2i\alpha}G^2(x,\theta,\bar\theta)\,. \end{aligned}$$

JHEP11(2004)071

In the massive vacuum, in which the massive S_{ij} obtains an expectation value and endows the matter in the fundamental with a mass through its quadratic term in the superpotential, the massive matter multiplets decouple from the gauge sector in the IR, and only the gauge part of the anomaly has to be matched in the perturbative superpotential. After matching, the order G^2 F-term is

$$\begin{aligned} W_1 \; = \; & \frac{N_f}{6}\Bigg[(2N_f-3)\log\left(-\frac{4\alpha}{m^2}S\right)-(2N_f-3)\log\left(1+\sqrt{1-\frac{4\alpha}{m^2}S}\right)- \\ & -(2N_f-1)\log\sqrt{1-\frac{4\alpha}{m^2}S}-2\log\frac{m}{\Lambda_1}-4N_c\log\lambda\Bigg]+\frac{1}{6}\log\frac{S}{\Lambda_1^3}\,. \end{aligned} \tag{3.20}$$

3.2 The vector model Ward identities

The partition function of the vector model corresponding to this model is

$$Z=\int [dQ_i^a][dS_{ij}]e^{-\frac{1}{g}W_{\rm tree}(Q,S_{ij})}\,, \tag{3.21}$$

where the action $W_{\rm tree}$ is given by (3.2). The coupling constant g is introduced for relating the vector model Ward identities with the gauge theory Konishi anomaly equations.

A transformation of the form $Q_a^i\to Q_a^i+\delta Q_a^i$ with S_{ij} unchanged generates the Ward identity

$$\left\langle \frac{\partial W_{\rm tree}}{\partial Q_a^i}\delta Q_a^i\right\rangle = g\left\langle \frac{\partial \delta Q_a^i}{\partial Q_a^i}\right\rangle\,. \tag{3.22}$$

Similarly, a transformation of the form $S_{ij}\to S_{ij}+\delta S_{ij}$ with Q_a^i fixed yields the Ward identity

$$\left\langle \frac{\partial W_{\rm tree}}{\partial S_{ij}}\delta S_{ij}\right\rangle = g\left\langle \frac{\partial \delta S_{ij}}{\partial S_{ij}}\right\rangle\,. \tag{3.23}$$

Using the same transformations leading to the gauge theory anomaly equations (3.3)–(3.9) we obtain the following Ward identities

$$2\lambda\left\langle S_{ij}M^{lj}\right\rangle = 2gN_c\delta_i^l\,, \tag{3.24}$$

$$\lambda\left\langle S_{lm}M^{ij}\right\rangle - mJ^{ij}\left\langle S_{lm}\right\rangle+2\alpha\left\langle S_{lm}S^{ij}\right\rangle = \frac{g}{2}\left(\delta_l^i\delta_m^j-\delta_m^i\delta_l^j\right)\,, \tag{3.25}$$

$$\begin{aligned} 2\lambda\left\langle S_{ij}M^{lj}M^n\right\rangle \; = \; & 2gN_c\left\langle M^n\right\rangle\delta_i^l+ \\ & +2ng(J^{-1})_{ij}\left\langle M^{lj}M^{n-1}\right\rangle\,, \end{aligned} \tag{3.26}$$

$$\begin{aligned} \lambda\left\langle S_{lm}M^{ij}\tilde{S}^n\right\rangle - mJ^{ij}\left\langle S_{lm}\tilde{S}^n\right\rangle + 2\alpha\left\langle S_{lm}S^{ij}\tilde{S}^n\right\rangle \; = \; & \frac{g}{2}\left\langle\tilde{S}^n\right\rangle\left(\delta_l^i\delta_m^j-\delta_m^i\delta_l^j\right)+ \\ & +ng\left\langle S_{lm}\tilde{S}^{n-1}\right\rangle J^{ij}\,, \end{aligned} \tag{3.27}$$

$$2\lambda\left\langle S_{ij}M^{lj}\tilde{S}^n\right\rangle = 2gN_c\left\langle\tilde{S}^n\right\rangle\delta_i^l\,, \tag{3.28}$$

$$\lambda\left\langle S_{lm}M^{ij}M^n\right\rangle - mJ^{ij}\left\langle S_{lm}M^n\right\rangle+2\alpha\left\langle S_{lm}S^{ij}M^n\right\rangle = \frac{g}{2}\left\langle M^n\right\rangle\left(\delta_l^i\delta_m^j-\delta_m^i\delta_l^j\right)\,, \tag{3.29}$$

$$2\lambda\left\langle S_{km}M^{ml}S_{il}\right\rangle = 2gN_c\left\langle S_{ki}\right\rangle\,. \tag{3.30}$$

Comparison of these equations with their gauge theory counterparts (3.3)–(3.9) yields that the gravitational genus one F-term is related to the vector model free energy by the relations

$$g = -\frac{1}{3}G^2\,, \qquad 2gN_c = S - \frac{2N_c}{3}G^2\,. \tag{3.31}$$

Hence, the contribution of planar diagrams to the perturbative part of the genus one F-term is given by a shift $S \to S - \frac{2N_c}{3}G^2$ [10], in the perturbative part of the effective superpotential of [9], taken about the massive vacuum

$$\begin{aligned} W_1^{\text{planar}} = -\frac{2N_c}{3}\frac{\partial W_{\text{eff}}^{\text{pert}}}{\partial S} = \frac{N_f N_c}{3}\Big[& -1 + 2\log\alpha - 2\log\left(m + \sqrt{m^2 - 4\alpha S}\right) - \\ & - 2\log\lambda\Big]\,, \end{aligned} \tag{3.32}$$

where the number of colors N_c has been specified explicitly. The perturbative part of W_1 proportional to N_c is

$$\frac{2N_f N_c}{3}\left[\log(-4\alpha S) - \log\left(m + \sqrt{m^2 - 4\alpha S}\right) - \log\lambda\right]$$

and all of it except the $\frac{2N_f N_c}{3}\log(-4S)$ term is accounted for by the planar contribution.

4. G_2 SYM with three flavors

This is an $\mathcal{N} = 1$ SYM theory with the gauge group G_2 with three flavors of chiral matter in the real fundamental **7** representation considered in [9]. The chiral superfields are denoted by Q_I^i (henceforth $i, j, k, \ldots = 1, \ldots, 7$ denote color indices and $I, J, K, \ldots = 1, \ldots, 3$ are flavor indices). The gauge invariant operators of this theory are the six mesons $X_{IJ} = \delta^{ij}Q_I^i Q_J^j$ and the single baryon $Z = \psi^{ijk}\epsilon_{IJK}Q_I^i Q_J^j Q_K^k$, where ψ^{ijk} is the G_2 invariant 3-form.

The tree-level superpotential

$$W_{\text{tree}} = m^{IJ}X_{IJ} + \lambda Z \tag{4.1}$$

is taken with the mass matrix $m^{IJ} = m\delta^{IJ}$, leaving the flavor symmetry intact.

4.1 Computation of W_1

4.1.1 The perturbative superpotential

The simplest Konishi equations for this theory (more explicit details are in appendix B)

$$2S\delta_{IJ} = 2m\langle X_{IJ}\rangle + \lambda\langle Z\rangle\delta_{IJ} + \frac{7}{3}G^2\delta_{IJ}\,, \tag{4.2}$$

$$0 = 2m\langle Z\rangle + 6\lambda\left(\langle X_I{}^I X_J{}^J\rangle - \langle X_{IJ}X^{IJ}\rangle\right), \tag{4.3}$$

$$\begin{aligned} 2S\langle \mathrm{tr}(X^n)\rangle\delta_{IL} = {} & 2m\langle X_{IL}\,\mathrm{tr}(X^n)\rangle + \lambda\langle Z\,\mathrm{tr}(X^n)\rangle\delta_{IL} + \frac{7}{3}G^2\langle\mathrm{tr}(X^n)\rangle\delta_{IL} + \\ & + \frac{2n}{3}G^2\langle (X^n)_{LI}\rangle\,, \end{aligned} \tag{4.4}$$

$$0 = 2m \left\langle Z \,\mathrm{tr}(X^n) \right\rangle + 6\lambda \left(\left\langle X_I{}^I X_J{}^J \,\mathrm{tr}(X^n) \right\rangle - \left\langle X_{IJ} X^{IJ} \,\mathrm{tr}(X^n) \right\rangle \right) + + \frac{2n}{9} G^2 \left\langle Z \,\mathrm{tr}(X^{n-1}) \right\rangle , \tag{4.5}$$

$$2S \left\langle X_J{}^I \right\rangle = 2m \left\langle X_{JK} X^{KI} \right\rangle + \lambda \left\langle X_J{}^I Z \right\rangle + \frac{1}{3} G^2 \left\langle X_K{}^K \right\rangle \delta_J{}^I + + \frac{8}{3} G^2 \left\langle X_J{}^I \right\rangle . \tag{4.6}$$

These equations can be solved and the needed correlation functions in the Higgsed vacuum found in [9] are

$$\left\langle X_I{}^I \right\rangle^g = -\frac{3m^3 - 72\lambda^2 S + 11\sqrt{m^6 - 36\lambda^2 m^3 S}}{4m\,(m^3 - 36\lambda^2 S)} , \tag{4.7}$$

$$\left\langle Z \right\rangle^g = -\frac{11m^3 - 432\lambda^2 S - 11\sqrt{m^6 - 36\lambda^2 m^3 S}}{6\lambda\,(m^3 - 36\lambda^2 S)} . \tag{4.8}$$

Utilizing the gradient equations (2.12) in this case

$$\frac{\partial W_1}{\partial m} = \left\langle X_I{}^I \right\rangle^g , \tag{4.9}$$

$$\frac{\partial W_1}{\partial \lambda} = \left\langle Z \right\rangle^g , \tag{4.10}$$

the order G^2 correction to the superpotential can be integrated in order to obtain

$$W_1 = -\frac{1}{12} \left[22 \log \left(1 + \sqrt{1 - \frac{36\lambda^2}{m^3} S} \right) + \log \left(1 - \frac{36\lambda^2}{m^3} S \right) + 42 \log m \right] + + C_1(S) . \tag{4.11}$$

4.1.2 The non-perturbative part

The non-perturbative part of W_1 is found as in [11] by requiring that in the limit $\lambda \to 0$

$$\Gamma_1 = \int d^4x d^2\theta W_1(S) G^2 \tag{4.12}$$

reproduce the $\mathrm{U}(1)_R$ anomaly.

The matter fields are all integrated out so we have to match only the anomaly in the gauge sector. Comparing with (24) in [11] and taking only terms of order G^2 we have in the gauge sector the anomaly

$$\mathcal{A} = -\frac{1}{3} G^2 (\mathrm{rank}) = -\frac{14}{3} G^2 , \tag{4.13}$$

since G_2 has 14 generators.

The $\mathrm{U}(1)_R$ transformation is defined by

$$\begin{aligned} \theta' &= e^{-i\alpha}\theta , \qquad \bar\theta' = e^{i\alpha}\bar\theta , \\ S'(x,\theta',\bar\theta') &= e^{-2i\alpha} S(x,\theta,\bar\theta) , \\ G'^2(x,\theta',\bar\theta') &= e^{-2i\alpha} G^2(x,\theta,\bar\theta) . \end{aligned} \tag{4.14}$$

The term $W_1^{\rm non-pert}G^2 = \frac{14}{6}G^2\log\frac{S}{\Lambda_1^3}$ has the required anomaly so we take $C_1(S) = \frac{7}{3}\log\frac{S}{\Lambda_1^3}$ and for solution (B.28) the correction becomes

$$\begin{aligned} W_1(S) \;=\; -\frac{1}{12}\Bigg[& 22\log\left(1+\sqrt{1-\frac{36\lambda^2}{m^3}S}\right)+\log\left(1-\frac{36\lambda^2}{m^3}S\right)+ \\ & +42\log m - 28\log\frac{S}{\Lambda_1^3}\Bigg]. \end{aligned} \tag{4.15}$$

In the $\lambda\to 0$ limit

$$\Gamma_1 \to \frac{1}{12}\int d^4x d^2\theta\left(28\log\frac{S}{\Lambda_1^3}-42\log m\right)G^2\,, \tag{4.16}$$

whose transformation is

$$\delta_{\mathrm{U}(1)_R}\Gamma_1 \to -\frac{14}{3}i\alpha G^2\,. \tag{4.17}$$

As argued in [11], the same scale has to be used throughout the G^2 term. Hence, dimensionality is taken care of in the expression

$$\begin{aligned} W_1 \;=\; -\frac{1}{12}\Bigg[& 22\log\left(1+\sqrt{1-\frac{36\lambda^2}{m^3}S}\right)+\log\left(1-\frac{36\lambda^2}{m^3}S\right)+ \\ & +42\log\frac{m}{\Lambda_1} - 28\log\frac{S}{\Lambda_1^3}\Bigg]. \end{aligned} \tag{4.18}$$

4.2 Comparison with the vector model

The partition function of the corresponding vector model is

$$Z = \int dQ e^{-\frac{1}{g}W_{\rm tree}(Q)}\,, \tag{4.19}$$

where the action is given by the tree-level superpotential of the gauge theory (4.1) and g is a coupling that should be replaced with a function of S and G^2 in order to reproduce the gauge theory anomaly equations.

In general, a transformation $Q_I^i \to Q_I^i + \delta Q_I^i$ generates the vector model Ward identity

$$\left\langle \frac{\partial W_{\rm tree}}{\partial Q_I^i}\delta Q_I^i\right\rangle = g\left\langle \mathrm{tr}\,\frac{\partial \delta Q_J^j}{\partial Q_I^i}\right\rangle\,. \tag{4.20}$$

The vector model Ward identity corresponding to the anomaly equation (4.2) is

$$2m\left\langle X_{IJ}\right\rangle + \lambda\left\langle Z\right\rangle\delta_{IJ} = \hat{7}g\delta_{IJ}\,, \tag{4.21}$$

where $\hat{7}$ denotes a factor of 7 coming from a trace on the $\mathbf{7}$ representation of G_2. The vector model counterpart of (4.3) is the Ward identity

$$2m\left\langle Z\right\rangle + 6\lambda\left(\left\langle X_I{}^I X_J{}^J\right\rangle - \left\langle X_{IJ}X^{IJ}\right\rangle\right) = 0\,. \tag{4.22}$$

This identity is actually identical to the anomaly equation. The transformation leading to (4.4) yields the Ward identity

$$2m \langle X_{IJ} \operatorname{tr}(X^n) \rangle + \lambda \langle Z \operatorname{tr}(X^n) \rangle \delta_{IJ} = \hat{7}g \langle \operatorname{tr}(X^n) \rangle \delta_{IJ} + 2ng \langle (X^n)_{IJ} \rangle \,. \tag{4.23}$$

Applying the same transformation as in (4.5) one obtains the vector model equation

$$2m \langle Z \operatorname{tr}(X^n) \rangle + 6\lambda \left(\langle X_I{}^I X_J{}^J \operatorname{tr}(X^n) \rangle - \langle X_{IJ} X^{IJ} \operatorname{tr}(X^n) \rangle \right) = \frac{2n}{3} g \left\langle Z \operatorname{tr}(X^{n-1}) \right\rangle \,. \tag{4.24}$$

Finally, the analog of (4.6) is

$$2m \left\langle X_{IK} X^K{}_J \right\rangle + \lambda \langle X_{IJ} Z \rangle = \hat{7}g \langle X_{IJ} \rangle + g \langle X_{IJ} \rangle + g \left\langle X_K{}^K \right\rangle \delta_{IJ} \,. \tag{4.25}$$

Comparison of the Ward identities (4.21)–(4.25) with the anomaly equation (4.2)–(4.6) yields the following identifications

$$g = -\frac{1}{3} G^2 \,, \qquad \hat{7}g = 2S - \frac{7}{3} G^2 \,. \tag{4.26}$$

The gauge group G_2 does not admit large-N_c expansion. Thus, it is not clear how to directly compare the gauge theory F-terms computation and the vector model diagrammatic expansion. The relation between the gauge theory anomaly equations and the vector model Ward identities suggests, that a method of comparison should exist. In particular, one may hope to identify which diagrams contribute to which F-term. We have given some details of the G_2 diagrammatics in appendix C. So far, we have not found a direct comparison scheme.

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A. Details for the DSB model

Using the Konishi transformation $\delta Q_a^i = \epsilon^i{}_j Q_a^j$, the following equation is obtained in the chiral ring

$$2\lambda \left\langle S_{ij} M^{lj} \right\rangle + \frac{2N_c}{3} G^2 \delta_i^l = S \delta_i^l \,. \tag{A.1}$$

The transformation $\delta S_{ij} = \epsilon_{ij}{}^{lm} S_{lm}$ leads to the equation

$$\lambda \left\langle S_{lm} M^{ij} \right\rangle - m \langle S_{lm} \rangle J^{ij} + 2\alpha \left\langle S_{lm} S^{ij} \right\rangle + \frac{1}{6} G^2 (\delta_l^i \delta_m^j - \delta_m^i \delta_l^j) = 0 \,. \tag{A.2}$$

The third transformation is $\delta Q_a^i = \epsilon^i{}_j Q_a^j M^n$, where $M \equiv (J^{-1})_{ij} M^{ij}$:

$$2\lambda \left\langle S_{ij} M^{lj} M^n \right\rangle + \frac{2N_c}{3} G^2 \langle M^n \rangle \delta_i^l + \frac{2n}{3} G^2 (J^{-1})_{ij} \left\langle M^{lj} M^{n-1} \right\rangle = S \langle M^n \rangle \delta_i^l \,. \tag{A.3}$$

Using $\delta S_{ij} = \epsilon_{ij}{}^{lm} S_{lm} \tilde{S}^n$, where $\tilde{S} \equiv J^{ij} S_{ij}$ one obtains

$$\begin{aligned}\lambda \left\langle S_{ij} M^{lm} \tilde{S}^n \right\rangle - m J^{lm} \left\langle S_{ij} \tilde{S}^n \right\rangle + 2\alpha \left\langle S_{ij} S^{lm} \tilde{S}^n \right\rangle + \\ + \frac{1}{6} G^2 \left\langle \tilde{S}^n \right\rangle (\delta^l_i \delta^m_j - \delta^l_j \delta^m_i) + \frac{n}{3} G^2 J^{lm} \left\langle S_{ij} \tilde{S}^{n-1} \right\rangle = 0 .\end{aligned} \tag{A.4}$$

The mixing of fundamental and gauge-singlet matter $\delta Q^i_a = \epsilon^i{}_l Q^l_a \tilde{S}^n$ yields

$$2\lambda \left\langle S_{ij} M^{lj} \tilde{S}^n \right\rangle + \frac{2N_c}{3} G^2 \left\langle \tilde{S}^n \right\rangle \delta^l_i = S \left\langle \tilde{S}^n \right\rangle \delta^l_i . \tag{A.5}$$

And with the transformation $\delta S_{ij} = \epsilon_{ij}{}^{lm} S_{lm} M^n$ one obtains the equation

$$\begin{aligned}\lambda \left\langle S_{ij} M^{lm} M^n \right\rangle - m J^{lm} \langle S_{ij} M^n \rangle + 2\alpha \left\langle S_{ij} S^{lm} M^n \right\rangle + \\ + \frac{1}{6} G^2 \langle M^n \rangle (\delta^l_i \delta^m_j - \delta^l_j \delta^m_i) = 0 .\end{aligned} \tag{A.6}$$

The transformation $\delta Q^i_a = \epsilon^i{}_j J^{kj} S_{kl} Q^l_a$ supplies us with the last equation we need

$$2\lambda \left\langle S_{kl} M^{lm} S_{im} \right\rangle + \frac{2N_c}{3} G^2 \langle S_{ki} \rangle = S \langle S_{ki} \rangle . \tag{A.7}$$

With the assumptions that connected three-point-functions vanish in the chiral ring, that connected two-point functions are proportional to G^2 and that the vacuum has flavor symmetry, the required correlation functions can be parameterized as follows

$$\left\langle M^{lm} \right\rangle = (M_0 + M_1 G^2) J^{lm} , \tag{A.8}$$

$$\langle S_{ij} \rangle = (S_0 + S_1 G^2)(J^{-1})_{ij} , \tag{A.9}$$

$$\left\langle \tilde{S} \right\rangle = -2N_f (S_0 + S_1 G^2) , \tag{A.10}$$

$$\left\langle S_{ij} M^{lm} \right\rangle_c = \left[A (J^{-1})_{ij} J^{lm} + B(\delta^l_i \delta^m_j - \delta^l_j \delta^m_i) \right] G^2 , \tag{A.11}$$

$$\langle S_{ij} M \rangle_c = 2(B - N_f A) G^2 (J^{-1})_{ij} , \tag{A.12}$$

$$\begin{aligned}\left\langle S_{ij} M^{lm} \right\rangle = {} & \left[S_0 M_0 + (S_0 M_1 + S_1 M_0 + A) G^2 \right] (J^{-1})_{ij} J^{lm} + \\ & + B G^2 (\delta^l_i \delta^m_j - \delta^l_j \delta^m_i) ,\end{aligned} \tag{A.13}$$

$$\left\langle S_{ij} S^{lm} \right\rangle_c = \left[C (J^{-1})_{ij} J^{lm} + D(\delta^l_i \delta^m_j - \delta^l_j \delta^m_i) \right] G^2 , \tag{A.14}$$

$$\begin{aligned}\left\langle S_{ij} S^{lm} \right\rangle = {} & \left[-S_0^2 + (C - 2S_0 S_1) G^2 \right] (J^{-1})_{ij} J^{lm} + \\ & + D G^2 (\delta^l_i \delta^m_j - \delta^m_i \delta^l_j) ,\end{aligned} \tag{A.15}$$

$$\left\langle M^{ij} M^{lm} \right\rangle_c = \left[E J^{ij} J^{lm} + F(J^{il} J^{jm} - J^{im} J^{jl}) \right] G^2 , \tag{A.16}$$

$$\left\langle M^{ij} M \right\rangle_c = -2(N_f E + F) G^2 J^{ij} , \tag{A.17}$$

$$\left\langle M^{ij} M \right\rangle = -2 \left[N_f M_0^2 + (2N_f M_0 M_1 + N_f E + F) G^2 \right] J^{ij} , \tag{A.18}$$

$$\begin{aligned}\left\langle S_{ij} M^{lj} M \right\rangle = {} & 2 \Big\{ N_f S_0 M_0^2 + \Big[N_f M_0 \left(2S_0 M_1 + S_1 M_0 - (2N_f - 1) B + 2A \right) - \\ & - M_0 B + S_0 (N_f E + F) \Big] G^2 \Big\} \delta^l_i ,\end{aligned} \tag{A.19}$$

$$
\begin{aligned}
\left\langle S_{ij}\tilde{S}\right\rangle_c &= 2(N_fC-D)G^2(J^{-1})_{ij}\,, && \text{(A.20)}\\
\left\langle S_{ij}\tilde{S}\right\rangle &= -2\left\{N_fS_0^2+\left[N_f\left(2S_0S_1-C\right)+D\right]G^2\right\}(J^{-1})_{ij}\,, && \text{(A.21)}\\
\left\langle M^{lm}\tilde{S}\right\rangle_c &= 2(B-N_fA)G^2J^{lm}\,, && \text{(A.22)}\\
\left\langle S_{ij}M^{lm}\tilde{S}\right\rangle &= -2\Big\{N_fS_0^2M_0+\Big[N_fS_0(S_0M_1+2S_1M_0+2A)- \\
&\qquad -M_0(N_fC-D)-S_0B\Big]G^2\Big\}(J^{-1})_{ij}J^{lm}- \\
&\quad -2N_fS_0BG^2(\delta_i^l\delta_j^m-\delta_j^l\delta_i^m)\,, && \text{(A.23)}\\
\left\langle S_{ij}S^{lm}\tilde{S}\right\rangle &= 2\left\{N_fS_0^3+\left[3N_fS_0\left(S_0S_1-C\right)+2S_0D\right]G^2\right\}(J^{-1})_{ij}J^{lm}- \\
&\quad -2N_fS_0DG^2(\delta_i^l\delta_j^m-\delta_j^l\delta_i^m)\,, && \text{(A.24)}\\
\left\langle S_{ij}M^{lm}M\right\rangle &= -2\Big\{N_fS_0M_0^2+\Big[N_fM_0(2S_0M_1+S_1M_0+2A)-M_0B+ \\
&\qquad +S_0(N_fE+F)\Big]G^2\Big\}(J^{-1})_{ij}J^{lm}- \\
&\quad -2N_fM_0BG^2(\delta_i^l\delta_j^m-\delta_j^l\delta_i^m)\,, && \text{(A.25)}\\
\left\langle S_{ij}S^{lm}M\right\rangle &= 2\Big\{N_fS_0^2M_0+\Big[N_fS_0(S_0M_1+2S_1M_0+2A)-N_fM_0C- && \text{(A.26)}\\
&\qquad -2S_0B\Big]G^2\Big\}(J^{-1})_{ij}J^{lm}-2N_fM_0DG^2(\delta_i^l\delta_j^m-\delta_j^l\delta_i^m)\,, \\
\left\langle S_{kl}M^{lm}S_{im}\right\rangle &= 2S_0[(2N_f-1)B-A]G^2(J^{-1})_{ki}+M_0[C-(2N_f-1)D]G^2(J^{-1})_{ki}+ \\
&\quad +[S_0^2M_0+(S_0^2M_1+2S_0S_1M_0)G^2](J^{-1})_{ik}\,. && \text{(A.27)}
\end{aligned}
$$

The Konishi anomaly equations are then expressed as the ten equations

$$
\begin{aligned}
2\lambda S_0M_0+S &= 0\,, && \text{(A.28)}\\
2\lambda\left[(2N_f-1)B-A-S_0M_1-S_1M_0\right]+\frac{2N_c}{3} &= 0\,, && \text{(A.29)}\\
(\lambda M_0-m-2\alpha S_0)S_0 &= 0\,, && \text{(A.30)}\\
\lambda(S_0M_1+S_1M_0+A)-mS_1+2\alpha(C-2S_0S_1) &= 0\,, && \text{(A.31)}\\
\lambda B+2\alpha D+\frac{1}{6} &= 0\,, && \text{(A.32)}\\
2\lambda[N_fM_0(2S_0M_1+S_1M_0-(2N_f-1)B+2A)-M_0B+S_0(N_fE+F)]- \\
-\frac{2N_fN_c+1}{3}M_0+N_fSM_1 &= 0\,, && \text{(A.33)}\\
-\lambda\left[N_fS_0(S_0M_1+2S_1M_0+2A)-M_0(N_fC-D)-S_0B\right]+ \\
+m\left[N_f(2S_0S_1-C)+D\right]+2\alpha\left[3N_fS_0(S_0S_1-C)+2S_0D\right] &= -\frac{S_0}{6}, && \text{(A.34)}\\
2\lambda[N_fS_0(S_0M_1+2S_1M_0+2A)-M_0(N_fC-D)- \\
-(N_f(2N_f-1)+1)S_0B]-\frac{2N_fN_c}{3}S_0+N_fSS_1 &= 0\,, && \text{(A.35)}\\
-\lambda\left[N_fM_0(2S_0M_1+S_1M_0+2A)-M_0B+S_0(N_fE+F)\right]+ \\
+m\left[N_f(S_0M_1+S_1M_0+A)-B\right]+ \\
+2\alpha\left[N_fS_0(S_0M_1+2S_1M_0+2A)-N_fM_0C-2S_0B\right] &= 0\,, && \text{(A.36)}
\end{aligned}
$$

$$4\lambda S_0\left[(2N_f-1)B-A\right]+2\lambda M_0\left[C-(2N_f-1)D\right]-$$
$$-2\lambda(S_0^2M_1+2S_0S_1M_0)+\frac{2N_c}{3}S_0 = SS_1\,, \quad \text{(A.37)}$$

whose solutions are

$$\begin{aligned}
S_0 &= \frac{-m+\sqrt{m^2-4\alpha S}}{4\alpha}\,,\\
S_1 &= \frac{4N_c\sqrt{m^2-4\alpha S}-(2N_f-1)(m+\sqrt{m^2-4\alpha S})}{12(m^2-4\alpha S)}\,,\\
M_0 &= \frac{m+\sqrt{m^2-4\alpha S}}{2\lambda}\,,\\
M_1 &= \frac{\alpha\left[4N_c\sqrt{m^2-4\alpha S}-(2N_f-1)(m+\sqrt{m^2-4\alpha S})\right]}{6\lambda(m^2-4\alpha S)}\,,\\
A &= 0\,,\\
B &= -\frac{m^2-4\alpha S+m\sqrt{m^2-4\alpha S}}{12\lambda(m^2-4\alpha S)}\,,\\
C &= 0\,,\\
D &= -\frac{m^2-4\alpha S-m\sqrt{m^2-4\alpha S}}{24\alpha(m^2-4\alpha S)}\,,\\
H &= \frac{\alpha(m^2-4\alpha S+m\sqrt{m^2-4\alpha S})}{6\lambda^2(m^2-4\alpha S)}\,,
\end{aligned} \quad \text{(A.38)}$$

and

$$\begin{aligned}
S_0 &= -\frac{m+\sqrt{m^2-4\alpha S}}{4\alpha}\,,\\
S_1 &= \frac{-4N_c\sqrt{m^2-4\alpha S}+(2N_f-1)(-m+\sqrt{m^2-4\alpha S})}{12(m^2-4\alpha S)}\,,\\
M_0 &= \frac{m-\sqrt{m^2-4\alpha S}}{2\lambda}\,,\\
M_1 &= \frac{\alpha\left[-4N_c\sqrt{m^2-4\alpha S}+(2N_f-1)(-m+\sqrt{m^2-4\alpha S})\right]}{6\lambda(m^2-4\alpha S)}\,,\\
A &= 0\,,\\
B &= -\frac{m^2-4\alpha S-m\sqrt{m^2-4\alpha S}}{12\lambda(m^2-4\alpha S)}\,,\\
C &= 0\,,\\
D &= -\frac{m^2-4\alpha S+m\sqrt{m^2-4\alpha S}}{24\alpha(m^2-4\alpha S)}\,,\\
H &= \frac{\alpha\left(m^2-4\alpha S-m\sqrt{m^2-4\alpha S}\right)}{6\lambda^2(m^2-4\alpha S)}\,,
\end{aligned} \quad \text{(A.39)}$$

where $H \equiv N_fE+F$. The solution (A.39) corresponds to the massive vacuum solution, in which the chiral multiplets are massive.

B. G_2 SYM with three flavors details

Using the most simple Konishi transformation, which involves only a flavor rotation $\delta Q^i_I = \lambda_I{}^J Q^i_J$, we obtain the Konishi equation

$$2m\left\langle X_{IJ}\right\rangle + \lambda\left\langle Z\right\rangle\delta_{IJ} + \frac{7}{3}G^2\delta_{IJ} = 2S\delta_{IJ}\,, \tag{B.1}$$

where the factor of two in front of the glueball superfield S is due to the representation index of the **7** representation of G_2. The transformation $\delta Q^i_I = \psi^{ijk}\epsilon_{IJK}Q^j_J Q^k_K$ generates the equation

$$2m\left\langle Z\right\rangle + 6\lambda\left(\left\langle X_I{}^I X_J{}^J\right\rangle - \left\langle X_{IJ}X^{IJ}\right\rangle\right) = 0\,. \tag{B.2}$$

A more general transformation is $\delta Q^i_M = \lambda_M{}^L Q^i_L\,\mathrm{tr}(X^n)$, where the trace is taken on flavor indices. The resulting equation is

$$\begin{aligned}2m\left\langle X_{IL}\,\mathrm{tr}(X^n)\right\rangle + \lambda\left\langle Z\,\mathrm{tr}(X^n)\right\rangle\delta_{IL} + \frac{7}{3}G^2\left\langle \mathrm{tr}(X^n)\right\rangle\delta_{IL} +&\\ +\frac{2n}{3}G^2\left\langle (X^n)_{LI}\right\rangle = 2S\left\langle \mathrm{tr}(X^n)\right\rangle\delta_{IL}\,.&\end{aligned} \tag{B.3}$$

A generalization of the second transformation, $\delta Q^i_I = \psi^{ijk}\epsilon_{IJK}Q^j_J Q^k_K\,\mathrm{tr}(X^n)$, yields for $n \geq 1$ the equation

$$\begin{aligned}2m\left\langle Z\,\mathrm{tr}(X^n)\right\rangle + 6\lambda\left(\left\langle X_I{}^I X_J{}^J\,\mathrm{tr}(X^n)\right\rangle - \left\langle X_{IJ}X^{IJ}\,\mathrm{tr}(X^n)\right\rangle\right) +&\\ +\frac{2n}{9}G^2\left\langle Z\,\mathrm{tr}(X^{n-1})\right\rangle = 0&\end{aligned} \tag{B.4}$$

The last required equation is obtained from $\delta Q^i_I = \lambda_I{}^J X_J{}^K Q^i_K$:

$$2m\left\langle X_{JK}X^{KI}\right\rangle + \lambda\left\langle X_J{}^I Z\right\rangle + \frac{1}{3}G^2\left\langle X_K{}^K\right\rangle\delta_J{}^I + \frac{8}{3}G^2\left\langle X_J{}^I\right\rangle = 2S\left\langle X_J{}^I\right\rangle\,. \tag{B.5}$$

If we assume that the vacuum does not break the SO(3) flavor symmetry and that connected two-point functions are of order G^2 ($\left\langle\sigma_I\sigma_J\right\rangle_c \sim G^2$), we may express the correlation functions in the form

$$\left\langle X_{IJ}\right\rangle = (A_0 + A_1G^2)\delta_{IJ}\,, \tag{B.6}$$
$$\left\langle Z\right\rangle = Z_0 + Z_1G^2\,, \tag{B.7}$$
$$\left\langle X_{IJ}X_{KL}\right\rangle_c = (B\delta_{IJ}\delta_{KL} + C\delta_{JK}\delta_{IL} + C\delta_{IK}\delta_{JL})G^2\,, \tag{B.8}$$
$$\left\langle X_{IJ}X_K{}^K\right\rangle_c = (3B + 2C)\delta_{IJ}G^2\,, \tag{B.9}$$
$$\left\langle ZX_{IJ}\right\rangle_c = D\delta_{IJ}G^2\,, \tag{B.10}$$
$$\left\langle X_I{}^I X_J{}^J\right\rangle_c = (9B + 6C)G^2\,, \tag{B.11}$$
$$\left\langle X_I{}^I X_J{}^J\right\rangle = 3\left[3A_0^2 + (3B + 2C + 6A_0A_1)\,G^2\right]\,, \tag{B.12}$$
$$\left\langle X_{IJ}X^{JI}\right\rangle = 3\left[A_0^2 + (B + 4C + 2A_0A_1)\,G^2\right]\,, \tag{B.13}$$
$$\left\langle X_{IJ}X_K{}^K\right\rangle = \left[3A_0^2 + (3B + 2C + 6A_0A_1)\,G^2\right]\delta_{IJ}\,, \tag{B.14}$$
$$\left\langle ZX_I{}^I\right\rangle = 3\left[A_0Z_0 + (D + A_0Z_1 + A_1Z_0)\,G^2\right]\,, \tag{B.15}$$
$$\left\langle X_{JK}X^{KI}\right\rangle = \left[A_0^2 + (2A_0A_1 + B + 4C)\,G^2\right]\delta_J{}^I\,, \tag{B.16}$$
$$\left\langle X_J{}^I Z\right\rangle = \left[A_0Z_0 + (A_0Z_1 + A_1Z_0 + D)\,G^2\right]\delta_J{}^I\,. \tag{B.17}$$

JHEP11(2004)071

By farther assuming that connected three-point functions vanish in the chiral ring one gets

$$\left\langle X_I{}^I X_J{}^J X_K{}^K \right\rangle = 27\left[A_0^3 + A_0\left(3B + 2C + 3A_0A_1\right)G^2\right], \tag{B.18}$$
$$\left\langle X_{IJ} X^{IJ} X_K{}^K \right\rangle = 3\left[3A_0^3 + A_0\left(9B + 16C + 9A_0A_1\right)G^2\right]. \tag{B.19}$$

Using the Konishi equations up to $n = 1$ yields the following equations for the seven unknowns A_0, A_1, Z_0, Z_1, B, C and D

$$2S = 2mA_0 + \lambda Z_0\,, \tag{B.20}$$
$$0 = 2mA_1 + \lambda Z_1 + \frac{7}{3}\,, \tag{B.21}$$
$$0 = mZ_0 + 18\lambda A_0^2\,, \tag{B.22}$$
$$0 = mZ_1 + 18\lambda\left(B - C + 2A_0A_1\right), \tag{B.23}$$
$$6SA_1 = 2m\left(3B + 2C + 6A_0A_1\right) + 3\lambda\left(D + A_0Z_1 + A_1Z_0\right) + \frac{23}{3}A_0\,, \tag{B.24}$$
$$0 = m\left(D + A_0Z_1 + A_1Z_0\right) + 6\lambda A_0\left(9B + C + 9A_0A_1\right) + \frac{1}{9}Z_0\,, \tag{B.25}$$
$$2SA_1 = 2m\left(2A_0A_1 + B + 4C\right) + \lambda\left(A_0Z_1 + A_1Z_0 + D\right) + \frac{11}{3}A_0\,, \tag{B.26}$$

whose two solutions are

$$\begin{aligned}
A_0 &= \frac{m^2 + m\sqrt{m^2 - 36\lambda^2 S/m}}{18\lambda^2}\,,\\
A_1 &= \frac{-3m^3 + 72\lambda^2 S + 11m^2\sqrt{m^2 - 36\lambda^2 S/m}}{12m^2\left(m^2 - 36\lambda^2 S/m\right)}\,,\\
Z_0 &= -\frac{m^3 - 18\lambda^2 S + m^2\sqrt{m^2 - 36\lambda^2 S/m}}{9\lambda^3}\,,\\
Z_1 &= -\frac{11m^3 - 432\lambda^2 S + 11m^2\sqrt{m^2 - 36\lambda^2 S/m}}{6\lambda m\left(m^2 - 36\lambda^2 S/m\right)}\,,\\
B &= \frac{\left(m + \sqrt{m^2 - 36\lambda^2 S/m}\right)^2}{108\lambda^2\sqrt{m^2 - 36\lambda^2 S/m}}\,,\\
C &= -\frac{m + \sqrt{m^2 - 36\lambda^2 S/m}}{108\lambda^2}\,,\\
D &= -\frac{m\left(m^2 - 18\lambda^2 S/m + m\sqrt{m^2 - 36\lambda^2 S/m}\right)}{27\lambda^3\sqrt{m^2 - 36\lambda^2 S/m}}\,,
\end{aligned} \tag{B.27}$$

and

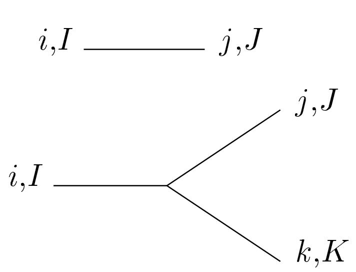
$$\begin{aligned}
A_0 &= \frac{m^2 - m\sqrt{m^2 - 36\lambda^2 S/m}}{18\lambda^2}\,,\\
A_1 &= -\frac{3m^3 - 72\lambda^2 S + 11m^2\sqrt{m^2 - 36\lambda^2 S/m}}{12m^2\left(m^2 - 36\lambda^2 S/m\right)}\,,\\
Z_0 &= \frac{-m^3 + 18\lambda^2 S + m^2\sqrt{m^2 - 36\lambda^2 S/m}}{9\lambda^3}\,,
\end{aligned}$$

$$
\begin{aligned}
Z_1 &= -\frac{11m^3 - 432\lambda^2 S - 11m^2\sqrt{m^2 - 36\lambda^2 S/m}}{6\lambda m\,(m^2 - 36\lambda^2 S/m)}\,,\\
B &= -\frac{\left(m - \sqrt{m^2 - 36\lambda^2 S/m}\right)^2}{108\lambda^2\sqrt{m^2 - 36\lambda^2 S/m}}\,,\\
C &= -\frac{m - \sqrt{m^2 - 36\lambda^2 S/m}}{108\lambda^2}\,,\\
D &= \frac{m^3 - 18\lambda^2 S - m^2\sqrt{m^2 - 36\lambda^2 S/m}}{27\lambda^3\sqrt{m^2 - 36\lambda^2 S/m}}\,.
\end{aligned}
\tag{B.28}
$$

Solution (B.28) corresponds to the Higgsed vacuum found in [9].

C. G_2 vector model diagrammatics

The Feynman rules, read from the G_2 vector model action, are



$$\frac{1}{2m} g \delta^{ij}\delta_{IJ}$$

$$-\frac{1}{g}\lambda\psi^{ijk}\epsilon_{IJK}$$

The vector model free-energy is defined by

$$e^{-\mathcal{F}(g)} = Z\,, \tag{C.1}$$

with Z the partition function given in (4.19). Using the rules and the identity

$$\psi^{ijk}\psi^{ilm} = \delta^{jl}\delta^{km} - \delta^{jm}\delta^{kl} + (*\psi)^{jklm}\,, \tag{C.2}$$

where $*\psi$ denotes the form dual to the 3-form ψ, the diagrams required for its perturbative computation (depicted in figure 1) can be computed:

$$
\begin{aligned}
&\frac{21}{2}\frac{1}{m}g \quad (a)\\
&\frac{189}{2}\frac{\lambda^2}{m^3}g \quad (b)\\
&\frac{15309}{4}\left(\frac{\lambda^2}{m^3}\right)^2 g^2 \quad (c)\\
-&\frac{5103}{8}\left(\frac{\lambda^2}{m^3}\right)^2 g^2 \quad (d)
\end{aligned}
$$

and the free-energy is

$$\mathcal{F}(g) = -\frac{21}{2}\frac{1}{m}g - \frac{189}{2}\frac{\lambda^2}{m^3}g - \frac{25515}{8}\left(\frac{\lambda^2}{m^3}\right)^2 g^2 + O(g^3)\,. \tag{C.3}$$

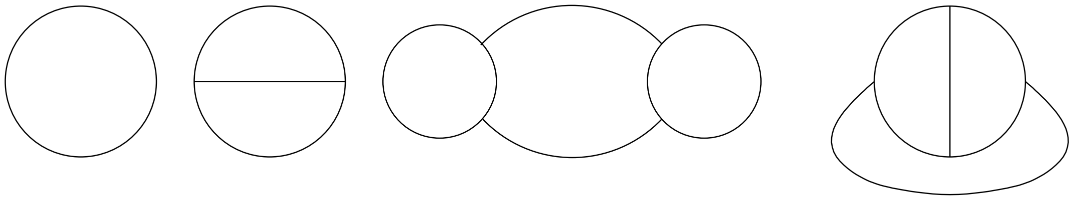


Figure 1: The G_2 free-energy diagrams.

If we *assume* that we may obtain the perturbative effective superpotential of the gauge theory by the identification $g = \frac{2}{7}S$ from the relation (4.26), we get under this assumption that the perturbative effective superpotential should be

$$W_{\text{eff}}^{\text{pert}} \stackrel{?}{=} -\frac{2}{7}S^2 \frac{\partial \mathcal{F}(\frac{2}{7}S)}{\partial S}\,. \tag{C.4}$$

This hypothesis is now tested using the perturbative result (C.3) and the exact effective superpotential [9]. The diagrammatic result is

$$\frac{6}{7}\frac{1}{m}S^2 + \frac{54}{7}\frac{\lambda^2}{m^3}S^2 + \frac{7290}{49}\left(\frac{\lambda^2}{m^3}\right)^2 S^3$$

while the expansion of the exact result is

$$W_{\text{eff}}^{\text{pert}} = (1 + \log 4 + 3\log m)S - 9\frac{\lambda^2}{m^3}S^2 - 81\left(\frac{\lambda^2}{m^3}\right)^2 S^3 + O(S^4)\,. \tag{C.5}$$

The above suggests that that even the lowest order diagram (figure 1*b*) has a gravitational contribution. Because of the structure of the vertex in this model there is no obvious way to identify "index loops" and — unlike gauge groups that admit large-N expansion — it does not have an expansion parameter that indicates the order of the gravitational F-terms, and the non-gravitational contribution do not seem to be diagrammatically isolated. Thus, it is conceivable, that this diagram includes both $(\text{tr}\, W_\alpha^2)^2 \sim S^2$ and $\text{tr}[(W_\alpha^2)^2] \sim SG^2$ terms with different coefficients.

References

[1] I. Antoniadis, E. Gava, K.S. Narain and T.R. Taylor, *Topological amplitudes in string theory*, *Nucl. Phys.* **B 413** (1994) 162 [hep-th/9307158].

[2] M. Bershadsky, S. Cecotti, H. Ooguri and C. Vafa, *Kodaira-Spencer theory of gravity and exact results for quantum string amplitudes*, *Commun. Math. Phys.* **165** (1994) 311 [hep-th/9309140].

[3] R. Dijkgraaf and C. Vafa, $N = 1$ *supersymmetry, deconstruction and bosonic gauge theories*, hep-th/0302011.

[4] H. Ita, H. Nieder, Y. Oz and T. Sakai, *Topological B-model, matrix models,* $\hat{c} = 1$ *strings and quiver gauge theories*, *J. High Energy Phys.* **05** (2004) 058 [hep-th/0403256].

JHEP11(2004)071

[5] H. Ooguri, A. Strominger and C. Vafa, *Black hole attractors and the topological string*, *Phys. Rev.* **D 70** (2004) 106007 [hep-th/0405146].

[6] R. Dijkgraaf and C. Vafa, *A perturbative window into non-perturbative physics*, hep-th/0208048.

[7] H. Ooguri and C. Vafa, *The c-deformation of gluino and non-planar diagrams*, *Adv. Theor. Math. Phys.* **7** (2003) 53 [hep-th/0302109].

[8] H. Ooguri and C. Vafa, *Gravity induced c-deformation*, *Adv. Theor. Math. Phys.* **7** (2004) 405 [hep-th/0303063].

[9] A. Brandhuber, H. Ita, H. Nieder, Y. Oz and C. Romelsberger, *Chiral rings, superpotentials and the vacuum structure of $N = 1$ supersymmetric gauge theories*, *Adv. Theor. Math. Phys.* **7** (2003) 269 [hep-th/0303001].

[10] J.R. David, E. Gava and K.S. Narain, *Konishi anomaly approach to gravitational F-terms*, *J. High Energy Phys.* **09** (2003) 043 [hep-th/0304227].

[11] H. Ita, H. Nieder and Y. Oz, *Gravitational F-terms of $N = 1$ supersymmetric* $\mathrm{SU}(N)$ *gauge theories*, *J. High Energy Phys.* **12** (2003) 046 [hep-th/0309041].

[12] A. Klemm, M. Marino and S. Theisen, *Gravitational corrections in supersymmetric gauge theory and matrix models*, *J. High Energy Phys.* **03** (2003) 051 [hep-th/0211216].

[13] R. Dijkgraaf, A. Sinkovics and M. Temurhan, *Matrix models and gravitational corrections*, *Adv. Theor. Math. Phys.* **7** (2004) 1155 [hep-th/0211241].

[14] L.F. Alday, M. Cirafici, J.R. David, E. Gava and K.S. Narain, *Gravitational F-terms through anomaly equations and deformed chiral rings*, *J. High Energy Phys.* **01** (2004) 001 [hep-th/0305217].

[15] L.F. Alday and M. Cirafici, *Gravitational F-terms of* SO/Sp *gauge theories and anomalies*, *J. High Energy Phys.* **09** (2003) 031 [hep-th/0306299].

[16] B.M. Gripaios, *Superpotentials for gauge and conformal supergravity backgrounds*, hep-th/0311025.

[17] H. Fuji and S. Mizoguchi, *Gravitational corrections for supersymmetric gauge theories with flavors via matrix models*, *Nucl. Phys.* **B 698** (2004) 53 [hep-th/0405128].

[18] F. Cachazo, M.R. Douglas, N. Seiberg and E. Witten, *Chiral rings and anomalies in supersymmetric gauge theory*, *J. High Energy Phys.* **12** (2002) 071 [hep-th/0211170].

[19] K. Konishi, *Anomalous supersymmetry transformation of some composite operators in SQCD*, *Phys. Lett.* **B 135** (1984) 439.

[20] K.-I. Konishi and K.-i. Shizuya, *Functional integral approach to chiral anomalies in supersymmetric gauge theories*, *Nuovo Cim.* **A90** (1985) 111.

[21] G. Veneziano and S. Yankielowicz, *An effective lagrangian for the pure $N = 1$ supersymmetric Yang-Mills theory*, *Phys. Lett.* **B 113** (1982) 231.

[22] K.-I. Izawa and T. Yanagida, *Dynamical supersymmetry breaking in vector-like gauge theories*, *Prog. Theor. Phys.* **95** (1996) 829 [hep-th/9602180].

[23] K.A. Intriligator and S. Thomas, *Dynamical supersymmetry breaking on quantum moduli spaces*, *Nucl. Phys.* **B 473** (1996) 121 [hep-th/9603158].

[24] D. Anselmi, D.Z. Freedman, M.T. Grisaru and A.A. Johansen, *Nonperturbative formulas for central functions of supersymmetric gauge theories*, *Nucl. Phys.* **B 526** (1998) 543 [hep-th/9708042].